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Mayer Expansions and the Hamilton–Jacobi Equation. II. Fermions, Dimensional Reduction Formulas

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We obtain estimates on effective actions for fermionic field theories by studying the flow of a continuous renormalization group transformation. For bosonic theories and statistical mechanics, we establish some new formulas for Mayer coefficients which are consequences of dimensional reduction.

KEY WORDS:

1. INTRODUCTION

We continue an investigation started in ref. 1, where the renormalization group, in the form of differential equations for the logarithm of a partition function, was used to derive estimates on connected correlation functions. These estimates were obtained by dominating the solution to the flow equation by a corresponding solution to a Hamilton–Jacobi equation. The limitation of this method was that the interaction was required to be bounded and analytic in a strip. In particular, it does not work for polynomial interactions.

In this paper we start to address this problem in two different ways. The first is that for purely fermionic field theories the anticommutation relations have the effect of making our requirements much less restrictive. The second is that for bosonic field theories "dimensional reduction" may offer a better way of dominating the flow equation by the Hamilton–Jacobi equation.

For fermions, it is well known that the exclusion principle is tantamount to the interaction being bounded, and, as a result, perturbation

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theory is convergent, provided it is organized by successive length scales. This observation was used in Ref. 2 to construct the Gross–Neveu model in the continuum. In Section 2 we use this boundedness to show that fermionic differential flow equations for the renormalization group can be written and solved by iteration in close analogy to ref. 1. Our estimates on the effective action resemble results in ref. 2, but our derivation should have some pedagogic value, as well as being more accurate. These estimates are intended to be used as bounds on the irrelevant parts of the effective action in studies of the renormalization group.

We now turn to bosonic theories and dimensional reduction. The flow equations in ref. 1 have the form

$$\frac{\partial V}{\partial t}=\frac{1}{2}\left[\Delta V-(\nabla V)^2\right], \qquad t>0 \tag{1.1}$$

We solved this equation by bounding its solution by the solution of a Hamilton–Jacobi equation

$$\frac{\partial \bar{V}}{\partial t}=+\frac{1}{2}(\nabla \bar{V})^2, \qquad t>0 \tag{1.2}$$

The defect in this procedure is that equation (1.2) has the wrong sign and develops singularities immediately if the initial data are given by $\lambda\varphi^4$, for example. If, on the other hand, we could relate (1.1) to the solution of the Hamilton–Jacobi equation

$$\frac{\partial \bar{V}}{\partial t}=-\frac{1}{2}(\nabla \bar{V})^2, \qquad t>0 \tag{1.3}$$

which has the same sign in its nonlinear part as (1.1), then we could hope to extend our approach to theories such as $\lambda\varphi^4$. The dimensional reduction formulas[(3)] express the solution of (1.1) as an integral (over initial data) of solutions to (1.3). We call this the dimensional reduction isomorphism. It is sometimes expressed in terms of integrals over classical actions, but this is the same thing, since the classical action is the solution to the Hamilton–Jacobi equation. Unfortunately, the dimensional reduction formulas are only known to hold to all orders in Feynman perturbation theory. Despite progress in ref. 4, there are as yet no cases other than Gaussian for which the complete isomorphism has been established nonperturbatively.[2] Indeed, the formulas have false implications for the Ising model.[(5)]

[2] See the remark below Eq. (1-5) of ref. 4.

In Section 3 we show that dimensional reduction also holds within the Mayer expansion. It implies some new formulas for the Mayer coefficients. This at least establishes the isomorphism for systems for which the Mayer expansion converges. To establish these formulas for field theories with more general interactions will require more work, but our present results may well be useful for the dipole gas[6] and the Kosterlitz–Thouless transition.

2. FERMIONS

We begin with a review of Berezin integration and Grassman algebras,[7] followed by the renormalization group and flow equations in this setting. Our main result is Theorem 2.3.

2.1. Grassman Algebras and Berezin Integration

Let $\mathscr{G}$ be the algebra over $\mathbb{C}$ whose generators are $\psi_1,\dots,\psi_{2g}$ with the anticommutation relations

$$\psi_i\psi_j+\psi_j\psi_i=0 \qquad \forall i,j=1,\dots,2g$$

We shall sometimes label the generators as $\psi_1,\dots,\psi_g,\psi_{\bar{1}},\dots,\psi_{\bar{g}}$. An element of this algebra has a unique expansion of the form

$$F=\sum_{N=0}^{2g}\frac{1}{N!}\sum_{i_1,\dots,i_N=1,\dots,2g} f_N(i_1,\dots,i_N)\,\psi_{i_1}\cdots\psi_{i_N}$$

where the $N=0$ term is understood to be a complex number f_0 and for each $N=1,2,\dots,2g$, f_N is antisymmetric.

We define partial derivatives $\partial/\partial\psi_i$ by demanding that $\partial\psi_j/\partial\psi_i=\delta_{ij}$, together with the requirements that $\partial/\partial\psi_i$ be linear over $\mathbb{C}$ and an antiderivation on $\mathscr{G}$, i.e.,

$$\frac{\partial}{\partial\psi_i}(AB)=\left(\frac{\partial}{\partial\psi_i}A\right)B+(-1)^{\text{degree}(A)}A\frac{\partial}{\partial\psi_i}(B)$$

for A and B monomials in ψ. Then we can rewrite F as a "Taylor series":

$$F(\psi)=\sum_{N=0}^{\infty}\frac{1}{N!}\sum_{i_1,\dots,i_N}\left(\frac{\partial}{\partial\psi_{i_N}}\cdots\frac{\partial}{\partial\psi_{i_1}}F\right)_{\psi=0}\psi_{i_1}\cdots\psi_{i_N}$$

Note that $\partial/\partial\psi_i$ and $\partial/\partial\psi_j$ anticommute. We can rewrite this more compactly:

$$F(\psi)=\sum_{I\subset\{1,\dots,2g\}}\partial^I F(0)\,\psi^I$$

Note that ∂^I is antiordered relative to Ψ^I. We shall fix the convention that Ψ^I is ordered with largest index to the right.

The Berezin integral of F is defined by

$$\int F = f(1, \bar{1}, 2, \bar{2}, ..., g, \bar{g})$$

Thus, $\int F$ is the top coefficient of F, with a convention on the order of indices to fix the sign.

Functions, in particular the exponential and the logarithm, are defined by their Taylor series about the $N=0$ term. Thus, if we write $F = f_0 + F_1$, where F_1 has vanishing $N=0$ term, then

$$e^F = e^{f_0} e^{F_1} = e^{f_0} \sum_{j=0}^{\infty} \frac{1}{j!} (F_1)^j$$

$$\log F = \log(f_0 + F_1) = \log f_0 + \log\left(1 + \frac{F_1}{f_0}\right)$$

$$= \log f_0 - \sum_{j=1}^{\infty} (-1)^j \frac{1}{j} \left(\frac{F_1}{f_0}\right)^j$$

One can only define $\log F$ if $f_0 \neq 0$. Both these series terminate after a finite number of terms because $F_1^p = 0$ if $p > 2g$, since by the anticommutation relations, we have $\psi_i^p = 0$ if $p = 2, 3, ...$.

2.2. Gaussian Integrals

For any $2g \times 2g$ matrix $\tilde{A}$ of the form

$$\tilde{A} = \begin{pmatrix} 0 & -A \\ A & 0 \end{pmatrix}$$

where A is a $g \times g$ symmetric and invertible matrix, we define the Gaussian Berezin integral $\int d\mu\, F$ by

$$\int d\mu\, F = \int [\exp(-1/2\psi\tilde{A}\psi)]\, F \Big/ \int \exp(-1/2\psi\tilde{A}\psi)$$

The denominator does not vanish, because

$$\int \exp(-1/2\psi\tilde{A}\psi) = \text{Pfaffian } \tilde{A}$$

$$= \det A \neq 0$$

Let $C = A^{-1}$ and $\tilde{C} = \tilde{A}^{-1}$, so that

$$\tilde{C} = \begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}$$

If we set $\xi = (\psi_1, ..., \psi_g)$ and $\bar{\xi} = (\psi_{\bar{1}}, ..., \psi_{\bar{g}})$, then $\frac{1}{2}\psi\tilde{A}\psi = \bar{\xi}A\xi$. This is the more standard type of notation.

The gaussian integral of $F \in \mathscr{G}$ can be evaluated using

$$\int d\mu\, \psi^S = \text{Pfaffian } \tilde{C}_S$$

$$= \det C_S$$

$\tilde{C}_S$ is the $|S| \times |S|$ matrix obtained from $\tilde{C}$ by deleting all rows and columns whose labels are not in S, and C_S is obtained by writing each element of S in the form i or $\bar{i} = i + g$, where $i = 1, ..., g$, and then deleting rows of C if they are not unbarred elements of S and columns if they are not barred elements of S. The dimension C_S is $\frac{1}{2}|S| \times \frac{1}{2}|S|$. If $|S|$ is not even or C_S is not a square matrix, $\int d\mu\, \psi^S = 0$.

Lemma 2.1. If C is positive, then

$$|\det C_S| \leqslant \max_{i \text{ or } \bar{i} \in S} C_{ii}^{|S|/2}$$

Proof. Suppose V, $\langle \cdot \rangle$, is a finite-dimensional inner product space. Gram's inequality states that if $f^{(1)}, ..., f^{(N)}$, $g^{(1)}, ..., g^{(N)} \in V$ and $M_{\alpha\beta} = \langle f^{(\alpha)}, g^{(\beta)} \rangle$, then

$$|\det M_{\alpha\beta}| \leqslant \left(\prod_{\alpha} \langle f^{(\alpha)}, f^{(\alpha)} \rangle \prod_{\beta} \langle g^{(\beta)}, g^{(\beta)} \rangle \right)^{1/2}$$

Let $\Lambda = \{i: i \text{ or } \bar{i} \in S\}$. Let $V = l^2(\Lambda)$ with inner product given by $\langle f, g \rangle = \sum_{i,j \in \Lambda} f_i (C + \varepsilon I)_{ij}\, g_j$. We let $\varepsilon > 0$ to make the inner product definite and take ε to zero at the end of the argument. Apply Gram's inequality with $f_i^{(\alpha)} = \delta_{i\alpha}$ and $g_j^{(\beta)} = \delta_{j\bar{\beta}}$, i.e., standard basis elements of $l^2(\Lambda)$, where $\alpha \in S$ and $\bar{\beta} \in S$. The lemma follows immediately. ∎

Proposition 2.2. Suppose $C = C^+ - C^-$, where C^+ and C^- are both positive and $\det C \neq 0$. Then

$$\left| \int d\mu\, \psi^S \right| \leqslant \left(4 \max_{\pm, i \text{ or } \bar{i} \in S} C_{ii}^{\pm}\right)^{|S|/2}$$

Proof. We write S as the union of two disjoint sets, $S = S_1 \cup S_2$, where S_1 contains only unbarred indices and S_2 contains only barred indices. Then

$$\left|\int d\mu\, \psi^S\right| = |\det(C^+ - C^-)_S|$$

$$= \left|\sum_{I \subset S_1} \sum_{J \subset S_2} \pm \det C^+_{I \times J} \det C^-_{(\overline{S_1 \backslash I}) \times (S_2 \backslash J)}\right|$$

$$\leqslant \sum_{I \subset S_1} \sum_{J \subset S_2} (\max C^{\pm}_{ii})^{|S|/2}$$

by Gram's inequality. We may assume that S contains equal numbers of barred and unbarred indices, because otherwise $\int d\mu\, \psi^S = 0$. Therefore, S_1 and S_2 each contain $|S|/2$ elements, and there are $2^{|S|/2}$ terms in each sum. ▮

2.3. Renormalization Group, Flow Equations

Suppose V is an even element of $\mathscr{G}$. We wish to study the following nonlinear map from the even subalgebra of $\mathscr{G}$ into itself, which we will call a renormalization group transformation:

$$V(\psi) \to (TV)(\psi) = -\log \int d\mu(\psi')\, e^{-V(\psi + \psi')}$$

$V(\psi + \psi')$ is an element of a Grassman algebra with $4g$ generators $\psi_1, \ldots, \psi_{2g}$ and $\psi'_1, \ldots, \psi'_{2g}$, but $d\mu(\psi')$ is a Gaussian integral just over one copy of $\mathscr{G}$ inside this larger algebra.

The map T depends on the $\tilde{A}$ and thus on C used to define the Gaussian measure. As in th Bose case, there is a semigroup property

$$T_{C_1 + C_2} = T_{C_1} \circ T_{C_2}$$

In view of this, it is natural to consider T as built up from infinitesimal transformations. If we introduce

$$C_{[t,s]} = \int_s^t \dot{C}(\tau)\, d\tau \tag{2.1}$$

where $\dot{C}$ is continuous in τ, then $T_{[s,u]} = T_{[s,t]} \circ T_{[t,u]}$ provided $u \leqslant t \leqslant s$ and each covariance is invertible.

If we define

$$V(t, \psi) = T_{[t,0]} V^{(0)}(\psi)$$

for some even $V^{(0)}$, then $V(t)$ satisfies

$$\frac{\partial V}{\partial t}=\frac{1}{2}\sum_{i,j}\tilde{\dot{C}}_{ij}(t)\left(\frac{\partial^2 V}{\partial\psi_i\partial\psi_j}-\frac{\partial V}{\partial\psi_i}\frac{\partial V}{\partial\psi_j}\right)$$
$$V(t=0)=V^{(0)} \tag{2.2}$$

The above flow equation is most easily derived by first proving that $F=\exp(-V)$ solves

$$\frac{\partial F}{\partial t}=\frac{1}{2}\sum_{i,j}\tilde{\dot{C}}_{ij}\frac{\partial^2 F}{\partial\psi_i\,\partial\psi_j}$$

using the definitions and integration by parts (ref. 7, 55) in analogy to ordinary heat equation. Then set $V=-\log F$ and use the "chain rule," which is valid for V in the even subalgebra of $\mathscr{G}$.

2.4. Norms

We measure the size of $\tilde{\dot{C}}$ by introducing

$$\|\tilde{\dot{C}}\|_1=\sup_i\sum_j|\tilde{\dot{C}}_{ij}|=\sup_i\sum_j \dot{C}_{ij}|$$
$$\sigma_{[t,s]}=\int_s^t d\tau\,[4\max_{i,\pm}\dot{C}^{\pm}_{ii}(\tau)]^{1/2} \tag{2.3}$$

The size of V is measured by

$$V_N=\frac{1}{N}\sup_i\sum_{\substack{I\ni i\\|I|=N}}|\partial^I V|_{\psi=0} \tag{2.4}$$

where $N=1, 2,\ldots$.

Theorem 2.3. Suppose $V^{(0)}$ is even and

$$v^{(0)}(\varphi)=\sum_{N=1}^{\infty}V_N^{(0)}\varphi^N \tag{2.5}$$

has a nonzero radius of convergence; then:

(i) For t and $|\varphi|$ small, $v^{(0)}(\varphi)$ may be extended to a function $v(t,\varphi)$ by solving

$$\frac{\partial v}{\partial t}=\dot{\sigma}_{[t,0]}\frac{\partial v}{\partial\varphi}+\frac{1}{2}\|\tilde{\dot{C}}(t)\|_1\left(\frac{\partial v}{\partial\varphi}\right)^2$$
$$v(0,\varphi)=v^{(0)}(\varphi) \tag{2.6}$$

For all t for which $v(t, \varphi)$ exists near $\varphi = 0$, the flow equations have a unique solution bounded according to

$$V_N(t) \leqslant v_N(t), N = 1, 2,... \tag{2.7}$$

where $v_N(t)$ is defined by

$$v(t, \varphi) = \sum_{N=0}^{\infty} v_N(t) \varphi^N \tag{2.8}$$

(ii) These bounds hold if

$$K\left[4 \int_0^t \|\dot{C}(\tau)\|_1 \, d\tau\right]^{1/2} + K\sigma_t < 1 \tag{2.9}$$

where

$$K = \sup_{N \geqslant 1} [NV(0)]^{1/N} \tag{2.10}$$

The principle underlying the proof is the same as in the Bose case in ref. 1. We convert the flow equation to a set of integral equations for the Taylor coefficients of V. The integral equations themselves have numerical coefficients which are Berezin integrals. These Berezin integrals are estimated using Proposition 2.2, leading to a majorant set of integral equations. We can then reverse the steps to go back to a flow equation in one ordinary variable φ.

Proof of Theorem, Part (i). If we substitute the Taylor series for V into both sides of the flow equation and equate coefficients of ψ^K for each K, the flow equation becomes a finite set of ordinary differential equations for the coefficients. Standard existence and uniqueness theorems apply and tell us the solution is unique.

To obtain bounds, we rewrite the flow equation as an integral equation

$$V(t, \psi) = \mu_{[t,0]} * V(0, \psi)$$
$$- \frac{1}{2} \int_0^t \sum_{i,j} \tilde{\dot{C}}_{ij}(s) \, \mu_{[t,s]} * \left(\frac{\partial V(s)}{\partial \psi_i} \frac{\partial V(s)}{\partial \psi_j}\right)$$

This implies, by differentiating both sides and setting ψ to zero, that

$$\partial^K V(t, \psi = 0) = \int d\mu_{[t,0]} \, \partial^K V(t = 0, \psi)$$

$$\pm \frac{1}{2} \int_0^t ds \sum_{i,j} \tilde{\dot{C}}_{ij}(s) \sum_{I \subset K} \int d\mu_{[t,s]} \, \partial^I \partial^i V(s, \psi) \, \partial^J \partial^j V(s, \psi)$$

where $J = K\backslash I$ and $\partial^i = \partial/\partial\psi_i$. The $\pm$ is determined by the subset $I \subset K$. (Recall that V is even.) We convert this to an equation for Taylor coefficients at $\psi = 0$ by substituting for $V(t, \psi)$ its Taylor expansion. We abbreviate $\partial^K V(t, \psi = 0)$ by $V_K(t)$. We have

$$\begin{aligned} V_K(t) = &\sum_{L \supset K} V_L(0) \int d\mu_{[t,0]}\, \partial^K \psi^L \\ &\pm \frac{1}{2} \int_0^t ds \sum_{i,j} \tilde{C}_{ij}(s) \sum_{I \subset K} \sum_{\substack{L \supset I \cup \{i\} \\ M \supset J \cup \{j\}}} V_L(s)\, V_M(s) \\ &\times \int d\mu_{[t,s]}\, (\partial^I \partial^i \psi^L\, \partial^J \partial^j \psi^M) \end{aligned} \tag{2.11}$$

This set of integral equations for the coefficients V_K generates a series by iteration.

If the series converges, it converges to the unique solution. By Proposition 2.2, the equations

$$\begin{aligned} W_K(t) = &\sum_{L \supset K} |V_L(0)|\, \sigma_{[t,0]}^{|L| - |K|} \\ &+ \frac{1}{2} \int_0^t ds \sum_{i,j} |\tilde{C}_{ij}(s)| \sum_{\substack{I \subset K - \{i\} \\ J \not\ni j}} \sum_{\substack{L \supset I \cup \{i\} \\ M \supset J \cup \{j\}}} \\ &\times W_L(s)\, W_M(s)\, \sigma_{[t,s]}^{|L| + |M| - |I| - |J| - 2} \end{aligned} \tag{2.12}$$

generate a majorant series, i.e.,

$$|W_K(t)| \geqslant |V_K(t)|$$

for all K, $|K| \geqslant 1$. We introduce

$$w(t, \varphi) = \sum_{N=1}^{\infty} N W_N(t)\, \varphi^{N-1}$$

$$v'(\varphi) = \sum_{N=1}^{\infty} N V_N(t=0)\, \varphi^{N-1}$$

where

$$W_N(t) = \frac{1}{N} \sup_k \sum_{\substack{K, K \ni k \\ |K| = N}} W_K(t)$$

and multiply both sides of (2.12) by $\varphi^{|K|-1}$. We then sum over all K

containing k, and then take the supremum over k. After a calculation given in the Appendix, we obtain, for $\varphi \geqslant 0$,

$$w(t, \varphi) \leqslant v'(\varphi + \sigma_{[t,0]}) + \frac{1}{2} \int_0^t ds \, \|\tilde{C}(s)\|_1 \frac{\partial}{\partial \varphi} w^2(s, \varphi + \sigma_{[t,s]}) \quad (2.13)$$

If we integrate this inequality and set $u(t, \varphi) = \int_0^\varphi d\varphi \, w(t, \varphi)$, it becomes

$$u(t, \varphi) \leqslant v(\varphi + \sigma_{[t,0]}) + \frac{1}{2} \int_0^t ds \, \|\tilde{C}(s)\|_1 \times \left(\frac{\partial u}{\partial \varphi}\right)^2 (s, \varphi + \sigma_{[t,s]}) \quad (2.14)$$

By its construction, the infinite series obtained by iterating (2.14), with the inequality replaced by equality, dominates the solution to the flow equation in the sense $V_N \leqslant u_N$, $N \geqslant 1$, where u_N is defined by

$$u(t, \varphi) = \sum_{N=0}^{\infty} u_N(t) \, \varphi^N$$

Since (2.14) corresponds to the differential equation

$$\frac{\partial u}{\partial t} = \dot{\sigma}_{[t,0]} \frac{\partial u}{\partial \varphi} + \frac{1}{2} \|\tilde{C}(t)\|_1 \left(\frac{\partial u}{\partial \varphi}\right)^2$$

$$u(t = 0) = v^{(0)}$$

we are finished with the proof of part (i).

Proof of Theorem, Part (ii). By definition of K, the initial data given by $v^{(0)}(\varphi)$ are majorized by

$$v(0, \varphi) = \sum_{N=1}^{\infty} \frac{1}{N} K^N \varphi^N = -\log(1 - K\varphi)$$

A short calculation shows that if we solve

$$\frac{\partial w}{\partial \tau} = \frac{1}{2} \left(\frac{\partial w}{\partial r}\right)^2; \qquad w(0, r) = -\log(1 - r) \quad (2.15)$$

and then define $v(t, \varphi)$ by

$$v(t, \varphi) = w(\tau, K(\varphi + \sigma_t)); \qquad \tau = \int_0^t K^2 \, \|\tilde{C}(s)\|_1 \, ds$$

then $v(t, \varphi)$ solves the equation in the theorem:

$$\frac{\partial v}{\partial t} = \dot{\sigma}_t \frac{\partial v}{\partial \varphi} + \frac{1}{2} \|\tilde{C}(t)\|_1 \left(\frac{\partial v}{\partial \varphi}\right)^2$$

$$v(0, \varphi) = -\log(1 - K\varphi)$$

The solution of (2.15) is

$$w(\tau, r) = -\log(1 - r_0) - \frac{1}{2\tau}(r - r_0)^2$$

where r_0 solves

$$\frac{1}{1 - r_0} + \frac{1}{\tau}(r - r_0) = 0$$

Therefore, w is analytic near $r = 0$ for $0 < \tau < 1/4$ and the radius of convergence is $1 - 2\sqrt{\tau}$. In order that v have a finite radius of convergence in φ, we need

$$1 - (4\tau)^{1/2} > K\sigma_t$$

which is the same as the condition of part (ii) of the theorem.

3. DIMENSIONAL REDUCTION

We shall prove dimensional reduction formulas for classical statistical mechanics in this section.

We begin by stating a theorem, Theorem 3.1, that says that dimensional reduction holds at each order in the Mayer expansion. A Mayer diagram is the "sum" of infinitely many Feynman diagrams, so this is not quite an immediate corollary of the results in the literature. Theorem 3.1 is an interesting formula for the Mayer coefficients, which is reminiscent of tree graph formulas already familiar in statistical mechanics.[8,1,9] As we shall argue, it is actually better, in that the tree graphs can be resummed to the classical action. For those systems for which the Mayer expansion converges, in particular, classical statistical mechanics with positive-definite short-range forces, a dimensional reduction formula for the logarithm of the partition function (Corollary 3.2) can then be proved, at least at low activity.

Let v_{ij}, $1 \leqslant i < j \leqslant N$, be arbitrary numbers. The standard identity used in high-temperature expansions is

$$\exp\left(-\sum v_{ij}\right) = \prod_{ij} [\exp(-v_{ij}) - 1 + 1] = \sum_G \prod_{ij \in G} [\exp(v_{ij}) - 1] \quad (3.1)$$

where G is summed over all subsets of

$$G_{\max} = \{ij: 1 \leqslant i < j \leqslant N\} \tag{3.2}$$

i.e., (Mayer) graphs on N vertices $\{1,..., N\}$. The connected part of $\exp(-\sum v_{ij})$ is defined by

$$\left[\exp\left(-\sum v_{ij}\right)\right]_c = \sum_{G \text{ connected}} \prod_{ij \in G} [\exp(-v_{ij}) - 1] \tag{3.3}$$

where G connected means G is a connected subgraph of $G_{\max}$ that meets all vertices. A *tree graph* T is defined to be a connected graph with no loops.

For each v_{ij} let there be given a C^1 function $\tilde{v}_{ij}(u)$, $u \geqslant 0$, with the following properties:

$$\begin{aligned} &(1) \quad \tilde{v}_{ij}(0) = v_{ij} && \text{all } ij \\ &(2) \quad \tilde{v}_{ij}(u) \to 0 && \text{as } u \to \infty \\ &(3) \quad \tilde{v}'_{ij} = \frac{\partial}{\partial u} \tilde{v}_{ij} && \text{is integrable} \end{aligned} \tag{3.4}$$

In the sequel, $\tilde{v}_{ij}$, $\tilde{v}'_{ij}$ will also denote functions on $\mathbb{R}^2 \times \mathbb{R}^2$ by setting

$$\tilde{v}_{ij} = \tilde{v}_{ij}((z_i - z_j)^2)$$

where $z_i, z_j \in \mathbb{R}^2$ and $(z_i - z_j)^2$ is the squared Euclidean distance between z_i and z_j.

Theorem 3.1.

(a) $$\left[\exp\left(-\sum v_{ij}\right)\right]_c = \left(\frac{1}{\pi}\right)^{N-1} \int d^{N-1}z \left(\sum_T \prod_{ij \in T} \tilde{v}'_{ij}\right) \exp\left(-\sum \tilde{v}_{kl}\right)$$

(b) $$= \pi \lim_{\Omega \uparrow \mathbb{R}^2} \frac{1}{|\Omega|} \int_\Omega \left(\frac{1}{\pi}\right)^N d^N z \times \left(\sum_T \prod_{ij \in T} \tilde{v}'_{ij}\right) \exp\left(-\sum \tilde{v}_{kl}\right)$$

where $\tilde{v}_{ij} = \tilde{v}_{ij}((z_i - z_j)^2)$ and $\tilde{v}'_{ij} = \tilde{v}'_{ij}((z_i - z_j)^2)$, and $\int d^{N-1} z$ is integration over $z_2,..., z_N \in \mathbb{R}^2$ with $z_1 = 0$; the limit as Ω increases is taken over Ω running through a sequence of spheres in $\mathbb{R}^2$ of increasing radius, $|\Omega|$ is the area of Ω, $\int_\Omega d^N z$ is integration over $z_1,..., z_N \in \Omega$, and T is summed over all tree graphs.

Note that the integrand on the right-hand side of (a) is integrable because

$$\int dz\, |\tilde{v}'_{ij}(z^2)| = \pi \int_0^\infty du\, |\tilde{v}'_{ij}(u)| < \infty \tag{3.5}$$

The proof of Theorem 3.1 is postponed until the end of the section.

We shall apply this theorem to the grand canonical partition function

$$Z = \sum_{N=0}^\infty \frac{\mathscr{Y}^N}{N!} \int d^N\mu\, e^{-V_N}$$

where

$$V_N = V_N(x_1, ..., x_N) = \frac{1}{2} \sum_{i,j} v(x_i, x_j)$$

and μ is a finite measure on a single-particle state space Λ. For simplicity, we shall assume

(Λ, μ) is a discrete finite space

(for example, a particle that can occupy one of finitely many sites in a lattice and have one of finitely many charges). These assumptions mean that $v(x, y)$ is a finite matrix. We assume

v is positive definite

$\tilde{v}$ is defined, using an operator notation in which v is a matrix operator and $\tilde{v}$ an integral operator with matrix indices, by

$$\tilde{v} = 4\pi(I \otimes v^{-1} + (-\Delta) \otimes I)^{-2} \tag{3.6}$$

or, more explicitly,

$$\begin{aligned} \tilde{v}(x, z, x', z') &= \frac{1}{\pi} \int dk\, e^{ik(z-z')}(k^2 + v^{-1})^{-2}(x, x') \\ &= \int_0^\infty dt\, e^{-(z-z')^2/4t}\, (e^{-tv^{-1}})(x, x') \end{aligned} \tag{3.7}$$

where functions of v are in the sense of operator calculus. The second representation is derived from the first using

$$A^{-2} = \int_0^\infty dt\, te^{-tA}$$

The second expression for $\tilde{v}$ enables us to check (3.4).[1-3] It shows that $\tilde{v}$ is a function of $(z-z')^2$. In particular, we can calculate $\tilde{v}'$. We will use the notation

$$\tilde{x} = (x, z)$$

We find

$$\begin{aligned}\tilde{v}'(\tilde{x}, \tilde{x}') &= \int_0^\infty dt \frac{1}{-4t} e^{-(z-z')^2/4t} (e^{-tv^{-1}})(x, x') \\ &= -\frac{1}{4\pi} \int dk \, (k^2 + v^{-1})^{-1}(x, x') \, e^{ik(z-z')} \end{aligned} \tag{3.8}$$

or, as operators,

$$\tilde{v}' = -\pi(-\Delta \otimes I + I \otimes v^{-1})^{-1} \tag{3.9}$$

Since $\tilde{v}$ is positive-definite and regular when $z \simeq z'$, we can construct a *Gaussian process* $\varphi(\tilde{x})$ with underlying probability measure dP such that φ is almost surely continuous in z and its covariance is $\tilde{v}$. Then

$$\int dP(\varphi) \exp\left[i \sum_{k=1}^N \varphi(\tilde{x}_k) \right] = \exp[-\tilde{V}_N(\tilde{x}_1, \ldots, \tilde{x}_N)] \tag{3.10}$$

where

$$\tilde{V}_N = \frac{1}{2} \sum_{i,j} \tilde{v}(\tilde{x}_i, \tilde{x}_j)$$

Therefore, Theorem 3.1 says

$$\begin{aligned}[\exp(-V_N)]_c = \lim_{|\Omega| \to \infty} \frac{\pi}{|\Omega|} \int dP(\varphi) \int_\Omega \left(\frac{1}{\pi}\right)^N d^N z \\ \times \left(\sum_T \prod_{ij \in T} \tilde{v}'_{ij} \right) \prod_1^N \exp[i\varphi(\tilde{x}_k)] \end{aligned} \tag{3.11}$$

Estimates in the proof of Theorem 3.1 show that the limit as $|\Omega| \to \infty$ is attained uniformly in φ, so we can interchange the limit and dP integral. We combine this with the Mayer expansion

$$\log Z = \sum \frac{\mathscr{Z}^N}{N!} \int d^N \mu \, (e^{-V_N})_c$$

and interchange the sum over N with the limit and the $dP(\varphi)$ integral to obtain

$$\log Z = \lim_{|\Omega| \to \infty} \frac{1}{|\Omega|} \int dP(\varphi)\, F(\Omega, \varphi)$$

where

$$\begin{aligned} F(\Omega, \varphi) &= \pi \sum \frac{\mathscr{Z}^N}{N!} \int_\Omega d^N\tilde{\mu} \sum_T \prod_{ij \in T} \tilde{v}'_{ij} \prod_{k=1}^{N} \exp[i\varphi(\tilde{x}_k)] \\ d\tilde{\mu}(\tilde{x}) &= \frac{1}{\pi}\, d\mu(x)\, dz \end{aligned} \tag{3.12}$$

The interchanges of sums are justified, along with the use of the high-temperature expansion, by the estimates

$$\int d^N\tilde{\mu} \prod_{ij \in T} |\tilde{v}'_{ij}| \leqslant C^N |\Omega|, \qquad \#\text{of trees} = N^{N-2} \tag{3.13}$$

which prove that the series for F converges absolutely if $|\mathscr{Z}|$ is small. The first estimate is easily derived[9] from the integrability of $\tilde{v}(\tilde{x}, \tilde{y})$ in $\tilde{y}$. The second is Cayley's theorem.

Finally, we note that the series for F is a sum of Feynman diagrams that are tree graphs and it is well known[10],3 that the sum of tree graphs is a power series in z for the corresponding classical action, which in this case is

$$S(\Omega, \varphi) = \frac{1}{2}\pi K(\varphi_0) + \mathscr{Z} \int d\mu \int_\Omega dz \exp\{i[\varphi_0(\tilde{x}) + \varphi(\tilde{x})]\} \tag{3.14}$$

where φ_0 is "classical field," i.e., the function $\varphi_0(\tilde{x})$ for which the right-hand side is critical, and $K(\varphi_0)$ is the bilinear form associated with the inverse of $-\tilde{v}'$. If we integrate the variational equation for φ_0 against the kernel $\tilde{v}'$, we obtain an integral equation. By solving this equation on $L_\infty(\Omega \times \cap$, $dz \times d\mu)$ using the contraction mapping principle, we find that φ_0 exists and is unique and analytic in $\mathscr{Z}$ for $|\mathscr{Z}|$ small. Therefore, $F(\Omega, \varphi)$ exists and equals the classical action. From (3.9)

$$\pi K(\varphi) = \int dz \sum_x [\nabla_z \varphi(\tilde{x})]^2 + \int dz \sum_{x,y} \varphi(\tilde{x})\, v^{-1}(x, y)\, \varphi(\tilde{y}) \tag{3.15}$$

In the second term $\tilde{x} = (x, z)$, $\tilde{y} = (y, z)$, i.e., the z coordinates are the same in $\tilde{x}$ and $\tilde{y}$.

We have proved the following theorem.

[3] See also ref. 1; cf. the remarks following Theorem 2.4.

Corollary 3.2. For $\mathscr{Z}$ sufficiently small

$$\log Z = \lim_{\Omega \uparrow \mathbb{R}^2} \frac{1}{|\Omega|} \int dP(\varphi) \left(\frac{\pi}{2} K(\varphi_0) \right.$$
$$\left. + \mathscr{Z} \int d\mu \int_\Omega dz \exp\{ i[\varphi_0(\tilde{x}) + \varphi(x)] \} \right)$$

where φ_0 is the critical point for the quantity in brackets.

Proof of Theorem 3.1. This proof is inspired by someηarguments in ref. 11.

It is enough to prove this theorem for $\tilde{v}$ an infinitely differentiable function with rapid decay for all derivatives at infinity, because we can approximate both sides by such functions.

Let $\theta_2,..., \theta_N$, $\bar{\theta}_2,..., \bar{\theta}_N$ generate a Grassman algebra with Berezin integral denoted by $\int d^{N-1}\theta\,(\cdot) = \int d\theta_2\, d\bar{\theta}_2 \cdots d\theta_N\, d\bar{\theta}_N\,(\cdot)$. (These ideas were reviewed in the last section.) Then

$$\left[\exp\left(-\sum v_{ij} \right) \right]_c = \left(-\frac{1}{\pi} \right)^{N-1} \int d^{N-1}z \int d^{N-1}\theta$$
$$\times \left\{ \exp\left[-\sum \tilde{v}_{ij}(z_{ij}^2 + \bar{\theta}_{ij}\theta_{ij}) \right] \right\}_c \tag{3.16}$$

where $z_{ij} = z_i - z_j$, $\theta_{ij} = \theta_i - \theta_j$, $z_1 = 0$, $\theta_1 = \bar{\theta}_1 = 0$. This is a consequence of the following lemma.

Lemma 3.3. Let $g(u_1,..., u_M)$ be infinitely differentiable and exponentially decaying at infinity; then,

$$\int d^{N-1}z \int d^{N-1}\theta\, g(u_1,..., u_M) = (-\pi)^{N-1}\, g(0)$$

where $u_1,..., u_M$ are arbitrary bilinears of the form

$$u(z, \theta) = \sum_2^N (z_i A_{ij} z_j + \bar{\theta}_i A_{ij} \theta_j)$$

with A_{ij} positive-definite.

Proof of Lemma 3.3. Substitute into the left-hand side

$$g(u_1,..., u_M) = \int d^M k\, \hat{g}(k_1,..., k_M) \exp[i(\mathbf{k} \cdot \mathbf{u})]$$

with each k_i, $i=1,\dots, M$, integrated along a contour that is an imaginary displacement of the real axis. This manifests decay in z, so that we can interchange the k integrals with the $\int d^{N-1}z \int d^{N-1}\theta$. Now use

$$\int d^{N-1}z \int d^{N-1}\theta \exp[i(\boldsymbol{k}\cdot\boldsymbol{u})] = \left\{\int d^{N-1}z \exp[i\mathbf{k}\cdot\mathbf{u}(z)]\right\} \times \left\{\int d^{N-1}\theta \exp[i\mathbf{k}\cdot\mathbf{u}(\theta)]\right\}$$

[where each component of $\mathbf{u}$ has the form $u(z)=\sum z_i A_{ij} z_j$, $u(\theta)=\sum \bar{\theta}_i A_{ij}\theta_j$]

$$\begin{aligned} &= (2\pi)^{N-1} \det{}^{-1}(-\mathbf{k}\cdot\mathbf{A}/2)\det(\mathbf{k}\cdot\mathbf{A}) \\ &= (-\pi)^{N-1} \end{aligned}$$

so

$$\begin{aligned} &\int d^{N-1}z \int d^{N-1}\theta\, g(u_1,\dots, u_M) \\ &\quad = (-\pi)^{N-1}\int d^M k\, \hat{g}(k_1,\dots, k_M) = (-\pi)^{N-1}\, g(0) \end{aligned}$$

which was to be proved. ∎

Proof of Theorem 3.1 (continued). Set $C=(-1/\pi)^{N-1}$. We have

$$\begin{aligned} &\left[\exp\left(-\sum v_{ij}\right)\right]_c \\ &\quad = C\int \left\{\exp\left[-\sum \tilde{v}_{ij}(z_{ij}^2+\bar{\theta}_{ij}\theta_{ij})\right]\right\}_c \\ &\quad = C \sum_{G \text{ connected}} \int \prod_{ij\in G} \{\exp[-\tilde{v}_{ij}(z_{ij}^2+\bar{\theta}_{ij}\theta_{ij})]-1\} \\ &\quad = C \sum_{G \text{ connected}} \sum_{H\subset G} \int \prod_{ij\in G\setminus H} \{\exp[-\tilde{v}_{ij}(z_{ij}^2)]-1\} \\ &\qquad \times \prod_{kl\in H} (\{\exp[-\tilde{v}_{kl}(z_{kl}^2)]\}[-\tilde{v}'_{kl}(z_{kl}^2)\,\bar{\theta}_{kl}\theta_{kl}]) \end{aligned} \tag{3.17}$$

In this last equality we expanded $\exp(-\tilde{v})-1$ in a Taylor series about z_{ij}^2, regarding $\bar{\theta}_{ij}\theta_{ij}$ as the perturbation. The Taylor series ends after two terms because $\theta_{ij}\theta_{ij}=0$. The Taylor expansion was followed by expanding the product:

$$\prod_{ij\in G}(A_{ij}+B_{ij})=\sum_{H\subset G}\prod_{ij\in G\setminus H}A_{ij}\prod_{kl\in H}B_{kl}$$

The Berezin integral $\int d^{N-1}\theta$ is now performed. First we note that $\int d^{N-1}\theta$ annihilates any term for which H is not a tree that connects all vertices. For example, if H contains a loop L, then the variables θ_{kl}, $kl\in L$, are not linearly independent (recall that $\theta_{kl}=\theta_k-\theta_l$), so θ_{kl}, $kl\in H$, are not linearly independent and

$$\int d^{N-1}\theta\prod_{kl\in H}\bar{\theta}_{kl}\theta_{kl}=0$$

If H has no loops, but fails to connect all the vertices, then H has strictly fewer than $N-1$ lines (the number of lines in a connected tree graph on $\{1,..., N\}$), and so

$$\int d^{N-1}\theta\prod_{kl\in H}\bar{\theta}_{kl}\theta_{kl}=0$$

because there is no $2(N-1)$-order term in the product. Thus in (3.17) we can assume that the sum over H is over connected tree graphs.

Now we claim that if H is a connected tree graph, then

$$\int d^{N-1}\theta\prod_{kl\in H}\bar{\theta}_{kl}\theta_{kl}=1$$

To see this, note that for $i=2,..., N$

$$\theta_i=\sum_{jk\in P(i)}\theta_{jk}$$

where $P(i)$ is the unique path in H that joins i to 1. By relabeling vertices, we can put the linear change of variables into a lower triangular form and see that the Jacobian is ± 1. When the same change of variables is applied to the $\bar{\theta}$'s, we obtain an overall Jacobian of 1. Then the claim follows from the Berezin change-of-variables formula.

These assertions, taken together, show that (3.17) becomes

$$\left[\exp\left(-\sum v_{ij}\right)\right]_c=C\sum_{G\text{ connected}}\sum_{T\subset G}\int d^{N-1}z$$

$$\times\prod_{ij\in G\setminus T}\{\exp[-\tilde{v}_{ij}(z_{ij}^2)]-1\}$$

$$\times\prod_{ij\in T}(\{\exp[-\tilde{v}_{ij}(z_{ij}^2)]\}[-\tilde{v}'_{ij}(z_{ij}^2)])$$

Now we interchange the sums over G and T and use

$$\sum_{G \supset T} \prod_{ij \in G \setminus T} [\exp(-\tilde{v}_{ij}) - 1] = \exp\left(-\sum_{ij \in G_{\max} \setminus T} \tilde{v}_{ij}\right)$$

which is (3.1) with $G_{\max}$ replaced by $G_{\max} \setminus T$. Note that if $G \supset T$, it is connected, so that a sum over all connected $G \supset T$ is an unconstrained sum over $G \setminus T$. Thus, we obtain

$$\left[\exp\left(-\sum v_{ij}\right)\right]_c = C \sum_T \int d^{N-1}z \prod_{ij \in G_{\max} \setminus T} \exp(-\tilde{v}_{ij}) \times \prod_{kl \in T} [\exp(-\tilde{v}_{kl})](-\tilde{v}'_{kl})$$

and when the products are combined, Theorem 3.1, part (a), is proved.

Proof of Part (b). By translation invariance in z we can average the result of part (a) over z_1 in some region Ω in $\mathbb{R}^2$ without changing it, i.e.,

$$\left[\exp\left(-\sum v_{ij}\right)\right]_c = \frac{1}{|\Omega|} \int_{z_1 \in \Omega} d^N z \, I$$

where

$$I = \left(\frac{1}{\pi}\right)^{N-1} \left(\sum_T \prod_{ij \in T} \tilde{v}'_{ij}\right) \exp\left(-\sum \tilde{v}_{kl}\right)$$

The difference between this and the result in part (b) is

$$D = \frac{1}{|\Omega|} \int_R d^N z \, I$$

where

$$R = \bigcup R_i, \qquad R_i = \{z_1 \in \Omega, z_i \in \mathbb{R}^2 \setminus \Omega\}$$

Divide Ω into a smaller concentric sphere Ω' and a shell S in such a way that as we take Ω increasing to all of $\mathbb{R}^2$, $|S|/|\Omega| \to 0$ and $d = \text{dist}(\Omega', \Omega^c) \to \infty$. We have

$$\frac{1}{|\Omega|} \int_{R_i} d^N z \, I \leqslant \frac{1}{|\Omega|} \int_S dz_1 \int d^{N-1} z \, I + \frac{1}{|\Omega|} \int_{\Omega'} dz_1 \int_{|z_2 - z_1| > d} d^{N-1} z \, I$$

The first term tends to zero as $|\Omega| \to \infty$ because $|S|/|\Omega| \to 0$. The second term goes to zero because it is less than

$$\int_{|z_2| > d} d^{N-1}z\, I(0, z_2, ..., z_N)$$

We have used the fact that I is integrable in $z_2, ..., z_N$ uniformly in Ω. ∎

APPENDIX. DERIVATION OF (2.13)

For the first term on the right-hand side of (2.12), (2.13), we have

$$\begin{aligned}\sum_{L \supset K} |V_L|\, \sigma^{|L|-|K|} &\to \sum_{K \ni k} \sum_{L \supset K} |V_L|\, \sigma^{|L|-|K|} \varphi^{|K|-1} \\ &= \sum_{L \ni k} |V_L| \sum_{\substack{K \ni k \\ K \subset L}} \sigma^{|L|-|K|} \varphi^{|K|-1} \\ &= \sum_{L \ni k} |V_L| \sum_{K' \subset L'} \sigma^{|L'|-|K'|} \varphi^{|K'|}\end{aligned}$$

[where $L' = L - \{k\}$, $K' = K - \{k\}$]

$$= \sum_{L \ni k} |V_L| (\varphi + \sigma)^{|L'|} \leqslant v'(\varphi + \sigma)$$

by the definition of v', since $|L'| = |L| - 1$.

For the second term we have to examine

$$\frac{1}{2} \int_0^t ds \sum_{i,j \neq k} |\tilde{C}_{ij}| \sum_{K \ni k} \sum_{I \subset K} \sum_{\substack{L \supset I \cup \{i\} \\ M \supset J \cup \{j\}}} W_L W_M \times \sigma^{|L| + |M| - 2 - |I| - |J|} \varphi^{|K|-1} \tag{A1}$$

There are additional constraints, $I \not\ni i$, $J \not\ni j$, $i, j \neq k$, and $J = K/I$, which we are not writing in, but which should be kept in mind. We recall that K is the disjoint union of I and J, so that

$$\begin{aligned}\sum_{K \ni k} \sum_{I \subset K} (\cdot) &= \sum_{I \ni k} \sum_{J \subset I^c} (\cdot) + \sum_{J \ni k} \sum_{I \subset J^c} (\cdot) \\ &\leqslant \sum_{I \ni k} \sum_{J} (\cdot) + \sum_{J \ni k} \sum_{I} (\cdot)\end{aligned} \tag{A2}$$

We substitute this into (A.1) and obtain two expressions corresponding to the two terms on the right of (A.2). The first is, after interchanging the J and M sums,

$$\frac{1}{2}\int ds \sum_{i,j} |\tilde{C}_{ij}| \sum_{I\ni k} \sum_{L\supset I\cup\{i\}} W_L \times \sum_{M\ni j} W_M \sum_{J\subset M} \sigma^{|L|+|M|-2-|I|-|J|}\varphi^{|I|+|J|-1}$$

We can do the sum over J using the binomial theorem

$$\sum_{J\subset M'} |\sigma|^{|M'|-|J|}\varphi^{|J|} = (\sigma+\varphi)^{|M'|}$$

with $M' = M - \{j\}$. Then we estimate the sum over M using the definition of $w(t, \varphi) = w(\varphi)$. The result is

$$\frac{1}{2}\int_0^t ds \sum_{i,j} |\tilde{C}_{ij}| \sum_{I\ni k} \sum_{L\supset I\cup\{i\}} W_L \sigma^{|L|-1-|I|}\varphi^{|I|-1} w(\varphi+\sigma)$$

Now do the sum over j,

$$\leqslant \frac{1}{2}\int ds \, \|\tilde{C}\|_1 \sum_i \sum_{I\ni k} \sum_{L\supset I\cup\{i\}} W_L \sigma^{|L|-1-|I|}\varphi^{|I|-1} w(\varphi+\sigma)$$

$$= \frac{1}{2}\int ds \, \|\tilde{C}\|_1 \sum_{L\ni k} W_L \times \sum_{i\in L-\{k\}} \sum_{\substack{I\ni k\\ I\subset L-\{i\}}} \sigma^{|L|-1-|I|}\varphi^{|I|-1} w(\varphi+\sigma)$$

(the sum over I is done using the binomial theorem, as above, then the sum over i)

$$= \frac{1}{2}\int ds \, \|\tilde{C}\|_1 \sum_{L\ni k} W_L(|L|-1)(\sigma+\varphi)^{|L|-2} w(\varphi+\sigma)$$

$$\leqslant \frac{1}{2}\int ds \, \|\tilde{C}\|_1 \left[\frac{\partial}{\partial\varphi} w(\varphi+\sigma)\right] w(\varphi+\sigma)$$

When we remember that this is half of (A.1) because we have an equal contribution from each term on the right of (A.2), we see that we have proven (2.13).

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